PROJECTIVE CONTACT MANIFOLDS

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ABSTRACT. The present work is concerned with the study of complex projective manifolds X which carry an additional complex contact structure. In the first part of the paper we show that if K_X is not nef, then either X is Fano and $b_2(X) = 1$, or X is of the form $\mathbb{P}(T_Y)$, where Y is a projective manifold. In the second part of the paper we consider contact manifolds where K_X is nef or, more generally, complex projective manifolds where the bundle of holomorphic 1-forms contains a nef sub-bundle of rank 1.

CONTENTS

1. INTRODUCTION

A compact complex manifold X of dimension $2n+1$ together with a subbundle $F \subset T_X$ of rank $2n$ is a *contact manifold* if the pairing $F \times F \to T_X/F =: L$ induced by the Lie bracket is everywhere non-degenerate. Alternatively, the non-degeneracy condition can be reformulated as follows: if $\theta \in H^0(X, \Omega^1_X \otimes L)$ is the induced form, then an elementary calculation shows that $\theta \wedge (d\theta)^{\wedge k}$ gives a section of $\Omega^{2k+1}_X \otimes L^k$. The subbundle $F = \ker(\theta)$ is non-degenerate in the above sense if $\theta \wedge (d\theta)^n$ has no zeroes.

Contact structures first came up in real geometry. One of the interests in complex geometry lies in the connection with twistor spaces and quaternionic Kähler manifolds. We refer to [Bea99] and [LeB95] for excellent introductions to these matters.

This paper contributes to the classification of projective contact manifolds. It has been proved by [Dru98] that $\kappa(X) = -\infty$. Therefore it seems natural to apply Mori theory for the classification.

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In section 2 we suppose that K_X is not nef, i.e. that there is a curve $C \subset X$ such that $K_X.C < 0$. The main result of this section is:

Theorem 1.1. *Let X be a projective contact manifold and assume that K_X is not nef. Then either X is Fano with $b_2 = 1$, or there exists a smooth projective manifold Y such that $(X, L) \cong (\mathbb{P}(T_Y), \mathcal{O}_{\mathbb{P}(T_Y)}(1))$.*

There are two further problems to be addressed for a complete classification. These are the following

Problem 1.2. It is necessary to classify the Fano contact manifolds with $b_2(X) = 1$.

It is conjectured that a Fano contact manifold with $b_2 = 1$ is a homogeneous variety which is the unique closed orbit in the projectivized (co)adjoint representation of a simple algebraic Lie group. The results of Beauville [Bea98] provide a strong evidence for this conjecture. The present work, however, is not concerned with this problem as we are primarily interested in those X which have second Betti-number > 1.

Problem 1.3. One has to show that K_X cannot be nef.

This should be a general consequence of $\kappa(X) = -\infty$; it is expected that manifolds with negative Kodaira dimension are uniruled (in dimension 3 this has already been shown by Miyaoka and Mori). However this conjecture is completely open in dimension at least 4.

In section 3 we prove that a hypothetical projective contact manifold X with K_X nef has $K_X^2 \equiv 0$. We present an approach that can help to rule out the case that K_X is nef: one knows that T_X is semi-stable with respect to the (degenerate) polarization (K_X, H^{m-2}), where $m = \dim X$ and H an arbitrary ample divisor. We would need to have semi-stability with respect to $(K_X + \epsilon H, H^{m-2})$ for some small $\epsilon > 0$.

This problem motivates us to consider a more general situation in section 4, namely that Ω^1_X contains a nef locally free subsheaf, $L^* \subset \Omega^1_X$ which is proportional to K_X. In this case we have $\kappa(X) \le 1$. If $\kappa(X) = 1$, then K_X is semi-ample and we obtain information on L^* via the Iitaka fibration. If $\kappa(X) = 0$ we conjecture that X is a quotient of a product $A \times Y$ where A is Abelian and Y is simply connected. We prove this conjecture if $\dim X \le 4$ or if Ueno's Conjecture K holds.

For the benefit of readers coming from differential geometry, we have added at various places some explanations concerning basic concepts of algebraic geometry, e.g. Mori theory.

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2. Contact manifolds where K_X is not nef

Notation 2.1. Let X be manifold of dimension $2n+1$ with a line bundle L. In this section the tensor product $\mathcal{F} \otimes L^{\otimes k}$ of an $\mathcal{O}_X$-sheaf $\mathcal{F}$ with a tensor power of L will be denoted by $\mathcal{F}(k)$ so that by $\Omega^p_X(k)$ we denote the sheaf of holomorphic p-forms on X twisted by $L^{\otimes k}$.

Let θ be a twisted holomorphic 1-form $\theta \in H^0(X, \Omega^1_X \otimes L) = H^0(X, \Omega^1_X(1))$.

For a trivializing covering $\{U_i\}$ in which the bundle L is given by transition functions g_{ij}, the form θ will be locally trivialized by $\theta_i \in \Omega^1_X(U_i)$ with the relation $\theta_i = g_{ij} \cdot \theta_j$. It is immediate to see that although usually $d\theta_i$ do not glue to a section of $\Omega^2_X(1)$, yet $\theta_i \wedge (d\theta_i)^n$ define a section of $\Omega^{2n+1}_X(n+1)$ which we call just $\theta \wedge (d\theta)^n$.

The manifold X is a *contact manifold* with the contact form $\theta \in H^0(X, \Omega^1_X(1))$ if $\theta \wedge (d\theta)^n$ does not vanish anywhere; this implies that

$$L^{\otimes(n+1)} \cong \det T_X = -K_X,$$

where K_X is the canonical bundle. The contact form θ defines a vector bundle epimorphism

$$T_X \to L \to 0.$$

We let F denote its kernel.

For an excellent introduction to contact manifolds we refer the reader to [LeB95].

2.1. **Preliminaries, $\mathbb{C}^*$-bundles and the Atiyah extension class.** First, let us discuss generalities related to a definition of the Chern class of a line bundle in terms of extensions of sheaves of differentials. This approach is due to Atiyah [Ati57]; we recall it and adjust it to our particular set-up. In the subsequent paragraphs X is an arbitrary complex manifold and L a line bundle on it.

Let $\pi : \mathbf{L}^\bullet \to X$ be the $\mathbb{C}^*$ bundle associated to L, as follows: on $\mathbf{L}^\bullet$ over U_i we have a coordinate $z_i \in \mathbb{C}^*$ and $z_i = g_{ij} z_j$. In other words $\mathbf{L}^\bullet = \mathrm{Spec}_X(\bigoplus_{k\in\mathbb{Z}} L^{\otimes k})$, or equivalently, $\mathbf{L}^\bullet$ is the total space of L^* with zero section removed. On $\mathbf{L}^\bullet$ we have a natural action of $\mathbb{C}^*$, that is: $\mathbb{C}^* \times \mathbf{L}^\bullet \ni (t, z) \mapsto t \cdot z \in \mathbf{L}^\bullet$ and related $\mathbb{C}^*$-invariant non-vanishing vector field $(z_i/\partial z_i)$ trivializes the relative tangent bundle $T_{\mathbf{L}^\bullet/X}$ which is the kernel of the tangential map $T\pi : T_{\mathbf{L}^\bullet} \to T_X$. Its dual $\Omega^1_{\mathbf{L}^\bullet/X}$ is thus trivialized by a $\mathbb{C}^*$-invariant form $\mu = dz_i/z_i$. So we have a short exact sequence

$$0 \to \pi^*(\Omega^1_X) \to \Omega^1_{\mathbf{L}^\bullet} \to \Omega^1_{\mathbf{L}^\bullet/X} \cong \mathcal{O}_{\mathbf{L}^\bullet} \to 0$$

which we can push down to X and use the fact $\pi_*(\mathcal{O}_{\mathbf{L}^\bullet}) = \bigoplus L^{\otimes k}$ to get

$$0 \to \bigoplus_{k\in\mathbb{Z}} \Omega^1_X(k) \to \pi_*(\Omega^1_{\mathbf{L}^\bullet}) \cong \bigoplus_{k\in\mathbb{Z}} (\Omega^1_{\mathbf{L}^\bullet})_k \to \bigoplus_{k\in\mathbb{Z}} L^{\otimes k} \to 0$$

where $\pi_*(\Omega^1_{\mathbf{L}^\bullet}) \cong \bigoplus (\Omega^1_{\mathbf{L}^\bullet})_k$ is the splitting into the weight spaces of the $\mathbb{C}^*$-action. We denote $(\Omega^1_{\mathbf{L}^\bullet})_0$ by $\mathcal{L}$ and thus find an exact sequence, called the Atiyah extension of L:

$$0 \to \Omega^1_X \to \mathcal{L} \to \mathcal{O}_X \to 0$$

Fact 2.2 (Atiyah). *The Chern class $c_1(L) \in H^1(X, \Omega^1_X) = \mathrm{Ext}^1_X(\mathcal{O}_X, \Omega^1_X)$ is the extension class of the above sequence.*

Remark 2.3. The sheaf $\mathcal{L}(1)$ may be identified as the sheaf of first jets of sections of L, see e.g. [BS95, 1.6.3], but we do not use this fact.

Fact 2.4. *If $f : Z \to X$ is a morphism then $c_1(f^*(L))$ is the image of $c_1(L)$ in the composition*

$$c_1(L) \in \mathrm{Ext}^1_X(\mathcal{O}_X, \Omega^1_X) \to \mathrm{Ext}^1_Z(\mathcal{O}_Z, f^*(\Omega^1_X)) \to \mathrm{Ext}^1_Z(\mathcal{O}_Z, \Omega^1_Z) \ni c_1(f^*(L))$$

with the first arrow coming from pull-back and the second one from the derivative $df : f^*(\Omega^1_X) \to \Omega^1_Z$. In particular, if $c_1(f^*(L))$ is non-zero then $f^*(\mathcal{L})$ is coming from a non-trivial extension in $\mathrm{Ext}^1_Z(\mathcal{O}_Z, f^*(\Omega^1_X))$.

2.1.1. *Projectivized Vector Bundles.* Let us consider a rank $r + 1$ vector bundle $\mathcal{E}$ over a smooth manifold Y. The projectivization of $\mathcal{E}$, denoted by $\mathbb{P}(\mathcal{E})$, is defined as the relative projective spectrum $Proj_X(Sym(\mathcal{E}))$ which geometrically means the variety of lines in the dual bundle $\mathcal{E}^*$. The variety $\mathbb{P}(\mathcal{E})$ admits a projection $p : \mathbb{P}(\mathcal{E}) \to Y$, with fibers being projective spaces $\mathbb{P}_r$, and a line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ whose restriction to each fiber is $\mathcal{O}_{\mathbb{P}_r}(1)$. Furthermore, we have $p_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{E}$. The cokernel of the differential map $dp : p^*(\Omega^1_Y) \to \Omega^1_{\mathbb{P}(\mathcal{E})}$ is $\Omega^1_{\mathbb{P}(\mathcal{E})/Y}$, the relative cotangent bundle.

The functoriality of the first Chern class implies the following lemma, whose proof we omit referring the reader to [Har77, II, 8.13] for a definition of the Euler sequence.

Lemma 2.5. *The image of the extension class* $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in \mathrm{Ext}^1_{\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, \Omega^1_{\mathbb{P}(\mathcal{E})})$ *under the induced map*

$$\mathrm{Ext}^1_{\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, \Omega^1_{\mathbb{P}(\mathcal{E})}) \to \mathrm{Ext}^1_{\mathbb{P}(\mathcal{E})}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, \Omega^1_{\mathbb{P}(\mathcal{E})/Y})$$

is associated with the (dual) relative Euler sequence

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/Y} \to p^*(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0.$$

2.2. **Symplectic structure on $\mathbf{L}^\bullet$.** For an introduction to these matters, see [LeB95, prop. 2.4]. Now we assume that X is a contact manifold with contact form θ. Then on $\mathbf{L}^\bullet$ we define a weight 1 homogeneous form $\omega \in H^0(X, (\Omega^2_{\mathbf{L}^\bullet})_1) \cong H^0(X, \wedge^2 \mathcal{L}(1))$ by setting:

$$\omega = d\theta + \theta \wedge \mu = d\theta_i + \theta_i \wedge (dz_i/z_i)$$

It is easy to verify that this gives a global 2-form on $\mathbf{L}^\bullet$ and moreover that $\omega^{n+1} = (n+1) \cdot \theta \wedge \mu \wedge (d\theta)^n$ so that the form ω is symplectic. Thus we obtain an isomorphism $\omega : \mathcal{L}^* \to \mathcal{L}(1)$ which we denote by the same name.

2.3. **Splitting type of T_X on rational curves.** It is a well-known fact attributed to Grothendieck that any vector bundle E on $\mathbb{P}_1$ decomposes into a sum of line bundles, $E \cong \bigoplus \mathcal{O}(e_i)$ where the (non-increasing) sequence of integers (e_i) is called the splitting type of E.

The main point in the proof of theorem 1.1 is an analysis of the splitting type of T_X, restricted to minimal rational curves. For this, we introduce the following

Notation 2.6. If E is a vector bundle on $\mathbb{P}_1$, let $\mathrm{rank}^+(E)$ denote the number of positive entries in the splitting type, that is, $\mathrm{rank}^+(E) = \#\{e_i > 0\}$.

We have the following elementary lemma:

Lemma 2.7. *Suppose that a bundle E is in an exact sequence of bundles on $\mathbb{P}_1$:*

$$0 \to E \to \bigoplus_{i=1}^{r} \mathcal{O}(a_i) \oplus \bigoplus_{j=1}^{s} \mathcal{O}(-b_j) \to \mathcal{O} \to 0$$

with $r \geq 0$, $s > 0$ and all $a_i \geq 0$, $b_j > 0$. If the sequence does not split, then $\mathrm{rank}^+(E^*) = s - 1$.

Proof. Since the sequence does not split, the map $H^0(E) \to H^0(\bigoplus_{i=1}^r \mathcal{O}(a_i))$ is an isomorphism and thus the number of non-negative values in the splitting type of E is r. Indeed, let $E^+ \hookrightarrow E$ be the sub-bundle associated to the non-negative values in the splitting type: then $H^0(E^+) = H^0(E) = H^0(\bigoplus \mathcal{O}(a_i))$ and the embedding $E^+ \hookrightarrow \bigoplus \mathcal{O}(a_i)$ is an isomorphism. Therefore the number of negative values in the splitting type of E is $s - 1$. ☐

Proposition 2.8. *Let $f : \mathbb{P}_1 \to C \subset X$ be a normalization of a rational curve in the contact manifold X. If $\deg(f^*(L)) = 1$, then $\mathrm{rank}^+(f^*(T_X)) = n$*

Proof. The symplectic form ω on $\mathbf{L}^\bullet$ gives an isomorphism $f^*(\mathcal{L}^*) \cong f^*(\mathcal{L}) \otimes \mathcal{O}(1)$, see [LeB95, prop. 2.4]. It follows that

$$f^*(\mathcal{L}) = \bigoplus_{i=1}^{n+1} (\mathcal{O}(a_i) \oplus \mathcal{O}(-a_i - 1))$$

where a_i are non-negative. On the other hand, by the preceding discussion on the Atiyah extension of L, it follows that the pullback of the sequence defining $\mathcal{L}$ to $\mathbb{P}_1$ does not split. Therefore we can apply the previous lemma and obtain that $\mathrm{rank}^+(f^*(T_X)) = n$. ☐

2.4. **Extremal rational curves on contact manifolds.** In the present section we apply proposition 2.8 to study the locus of minimal rational curves passing through a given point of X. We use Kollár's book [Kol96] as our main reference.

Let us consider the scheme $\mathrm{Hom}(\mathbb{P}_1, X)$ parameterizing morphisms from $\mathbb{P}_1$ to X. By $\mathrm{Hom}(\mathbb{P}_1, X; 0 \mapsto x)$ we denote the scheme parameterizing morphisms sending $0 \in \mathbb{P}_1$ to $x \in X$. Let $F : \mathbb{P}_1 \times \mathrm{Hom}(\mathbb{P}_1, X) \to X$ be the evaluation morphism. The restriction of F to any family of morphisms $V \subset \mathrm{Hom}(\mathbb{P}_1, X)$ will be denoted by F_V. We write $\mathrm{locus}(V)$ for the image of the evaluation $F_V(\mathbb{P}_1 \times V)$. By abuse, we will say that V is a family of rational curves and $\mathrm{locus}(V)$ is their locus. If $x \in \mathrm{locus}(V)$ is any point, then let $\mathrm{locus}(V, x)$ denote $\mathrm{locus}(V \cap \mathrm{Hom}(\mathbb{P}_1, X; 0 \mapsto x))$, that is, the locus of curves from V which contain x.

Following Kollár, we say that the family V is *unsplit* if curves from the family can not be deformed to a 1-cycle consisting of more than one (counting possible multiplicities) curves —for a precise definition we refer to [Kol96, IV, 2.1]. We note that in our case V is unsplit if $f(\mathbb{P}_1)$ is an extremal rational curve in the sense of Mori (see the definition below).

Proposition 2.9. *Let X be a projective contact manifold of dimension $2n + 1$ and let V be an irreducible component of $\mathrm{Hom}(\mathbb{P}_1, X)$. If V is an unsplit family of rational curves and for $f \in V$ we have $\deg(f^*(L)) = 1$, then $\mathrm{locus}(V) = X$ and moreover $\dim \mathrm{locus}(V, x) = n$ for any $x \in X$.*

Proof. Let $f : \mathbb{P}_1 \to X$ be a morphism such that $[f] \in V$ and $f(0) = x$. By [Kol96, II 3.10] and proposition 2.8 the tangential map TF satisfies

$$\mathrm{rank}\left[TF_{(p,[f])}\right] = \mathrm{rank}^+ f^*(T_X) = n$$

for any $p \in \mathbb{P}_1 - \{0\}$. Therefore, by [Har77, III, 10.6], for *any* $x \in \mathrm{locus}(V)$ we have $\dim \mathrm{locus}(V, x) \leq n$. On the other hand, by Kollár's version of the Ionescu-Wiśniewski estimate [Kol96, IV, 2.6.1] for any $x \in \mathrm{locus}(V)$ we have

$$\dim X + \deg(f^*(-K_X)) = 3n + 2 \leq \dim \mathrm{locus}(V) + \dim \mathrm{locus}(V, x) + 1$$

Note that the quoted inequality [Kol96, IV, 2.6.1]) is stated for a *general* point x in the locus of a *generically* unsplit family V. However we claim that it remains true for *any* $x \in \mathrm{locus}(V)$ if the family is unsplit. The proof is the same as the one of [Kol96, prop. IV, 2.5] —we point out that there is an obvious misprint reversing the inequality in this proposition. Now comparing the inequalities we get:

$$3n + 2 \leq \dim \mathrm{locus}(V) + \dim \mathrm{locus}(V, x) + 1 \leq \dim X + n + 1 = 3n + 2$$

so that $\mathrm{locus}(V) = X$ and $\dim \mathrm{locus}(V, x) = n$ for any $x \in X$. □

2.5. **Contractions of contact manifolds. Proof of theorem 1.1.** For the convenience of the reader we recall basic facts of Mori theory before commencing the proof of theorem 1.1. For a complete treatment of this subject we refer to e.g. [KM98].

2.5.1. *Mori Theory.* Let X be a complex projective manifold such that K_X is not nef. This means that there exists a curve C such that $K_X \cdot C < 0$. Then, by the Kawamata-Shokurov base-point-free theorem, X admits a Mori contraction. That is, there exists a normal projective variety Y and a surjective morphism $\phi : X \to Y$ (which is not an isomorphism) such that $\phi_*(\mathcal{O}_X) = \mathcal{O}_Y$ (i.e. its fibers are connected) and $-K_X$ is ϕ-ample.

The contraction ϕ is called *elementary* if all curves contracted by ϕ to points are numerically proportional, or equivalently, if $b_2(X) = b_2(Y) + 1$. It is a basic fact that any contraction can be factored via an elementary one.

If ϕ is an elementary contraction, then we define its *length* $l(\phi) := \min\{-K_X \cdot C\}$, where C is among rational curves contracted by ϕ (a rational curve C for which the minimum is achieved we call an extremal rational curve). According to Mori's cone theorem, the number $l(\phi)$ is defined (i.e. there exists a rational curve contracted by ϕ) and $\dim X + 1 \geq l(\phi) \geq 1$.

If X is a projective contact manifold then, since $-K_X = (n+1)L$, the bundle L is ϕ-ample for any Mori contraction ϕ of X. Moreover, if ϕ is elementary then $l(\phi)$ is either $2n+2$ or $n+1$, and the latter occurs if there exists a rational curve C contracted by ϕ such that $L \cdot C = 1$. If $l(\phi) = 2n+2 = \dim X + 1$ then, by a result of Ionescu, see e.g. [Kol96, IV, 2.6.1], the contraction ϕ is onto a point and therefore X is Fano with $b_2 = 1$.

2.5.2. *Proof of theorem 1.1.* In view of the theory outlined in the previous paragraph, in order to prove theorem 1.1, it remains to show that an elementary contraction $\phi : X \to Y$ is always isomorphic to $\mathbb{P}(T_Y) \to Y$.

As a first step, we show that in our setup Mori contractions cannot be birational.

Lemma 2.10. *Let $\phi : X \to Y$ be a Mori contraction of a projective contact manifold X. Then $\dim Y < \dim X$.*

Proof. Since any Mori contraction can be factored through an elementary one we may assume that ϕ is elementary and moreover, by the previous considerations, that $l(\phi) = n+1$, because otherwise Y is a point. Let $C \subset X$ be a rational curve contracted by ϕ such that $-K_X \cdot C = (n+1) \cdot L \cdot C = n+1$. Let $f : \mathbb{P}_1 \to C$ be a normalization of C and consider an irreducible component V of $\mathrm{Hom}(\mathbb{P}_1, X)$ which contains $[f]$. By proposition 2.9, $\mathrm{locus}(V) = X$ so that the lemma follows. □

Proposition 2.11. *Let X be as above and $\phi : X \to Y$ be a surjective morphism to a variety where $0 < \dim Y < \dim X$. If X_η is a generic fiber and X_η is Fano, then $X_\eta \cong \mathbb{P}_n$ and X_η is an integral manifold of F, i.e. $T_{X_\eta} \subset F$.*

Proof. We may assume that X_η and $\phi(X_\eta)$ are smooth and that ϕ has maximal rank at every point of X_η.

As a first step we show that $X_\eta \cong \mathbb{P}_n$. In order to do this, construct a sheaf-morphism $\beta : L|_{X_\eta} \to T_{X_\eta}$. Take $Hom(.,L)$ of the sequence defining the contact structure and obtain

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \otimes L \longrightarrow F^* \otimes L \longrightarrow 0.$$

Contraction with the contact form $\omega \in H^0(X, (F \otimes F)^* \otimes L)$ yields a morphism $\iota\omega : F \to F^* \otimes L$. Since ω is non-degenerate, this must be an isomorphism $F \cong F^* \otimes L$. Thus, one obtains a map $\alpha : \Omega^1_X \otimes L \to T_X$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega^1_X \otimes L & \longrightarrow & F^* \otimes L & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longleftarrow & L & \longleftarrow & T_X & \longleftarrow & F & \longleftarrow & 0 \end{array} \tag{2.1}$$

Now consider the canonically defined map $\phi^*(\Omega^1_Y) \otimes L \to \Omega^1_X \otimes L$. Restrict this map to X_η and note that $\phi^*(\Omega^1_Y)|_{X_\eta} \cong \oplus^{\dim Y} \mathcal{O}_{X_\eta}$. This yields a sequence of morphisms as follows:

$$\begin{tikzcd} & & & T_{X_\eta} \arrow[d] \\ (\phi^*(\Omega^1_Y) \otimes L)|_{X_\eta} \cong (L^{\oplus \dim Y})|_{X_\eta} \arrow[urrr, dashed, "\beta"] \arrow[r] & (\Omega^1_X \otimes L)|_{X_\eta} \arrow[rr, "\alpha|_{X_\eta}"] & & T_X|_{X_\eta} \end{tikzcd}$$

The induced map β exists because the normal bundle N_{X_η} is trivial, but $L|_{X_\eta}$ is ample so that every morphism $L \to N_{X_\eta}$ is necessarily trivial.

Now claim that β is not the trivial map. Since ϕ has maximal rank at some point of X_η, the map $(L^{\oplus m})|_{X_\eta} \to (\Omega^1_X \otimes L)|_F$ is not identically zero. On the other hand, diagram 2.1 implies directly that $Ker(\alpha) \cong \mathcal{O}_X$. Again using that there is no map $L|_{X_\eta} \to \mathcal{O}_{X_\eta}$, we see that β is non-trivial indeed.

The existence of β is equivalent to $h^0(X_\eta, T_{X_\eta} \otimes L^*|_{X_\eta}) \geq 1$. In this situation a theorem of J. Wahl [Wah83] applies, showing that $X_\eta \cong \mathbb{P}_k$ for some $k \in \mathbb{N}$. Use the adjunction formula to see that $k = n$.

In order to see that X_η is F-integral, let $x \in X_\eta$ be any point and note that the tangent space $T_{X_\eta}|_{\{x\}}$ is spanned by the tangent space of the lines in X_η passing through x. Thus, in order to show that X_η is integral, it is sufficient to show that all lines in X_η are integral manifolds. If $C \subset X_\eta$ is a line, using the adjunction formula, we obtain $-K_X.C = n+1$, i.e. $L.C = 1$. If $\iota : C \to X$ is the embedding, note that, since $T_C \cong \mathcal{O}_C(2)$, the induced map $T_C \to L$ is necessarily trivial. This shows that $T_{X_\eta} \subset F|_{X_\eta}$. □

The preceding proposition can also be proved using the Atiyah extension class of L and the symplectic form ω, c.f. (2.2). We may now finish with the proof of theorem 1.1.

Theorem 2.12. *Let $\phi : X \to Y$ be a Mori contraction of a projective contact manifold X of dimension $2n+1$ onto a positive dimensional variety Y. Then Y is smooth of dimension $n+1$, moreover $X \cong \mathbb{P}(T_Y)$ and $L \cong \mathcal{O}_{\mathbb{P}(T_Y)}(1)$.*

Proof. First we prove that Y is smooth and $X = \mathbb{P}(\phi_*(L))$. This will follow from a result by Fujita [Fuj85, 2.12], if only we prove that the morphism ϕ is equidimensional (i.e. it has all fiber of dimension n).

After twisting L by a pull-back of an ample line bundle from Y we may assume that the result, call it L', is ample. Let $\mathrm{Chow}_{n,1}(X)$ be a Chow variety of n-dimensional cycles on X of degree 1 with respect to L', for the definition see e.g. [Kol96, I, 3]. Let X_g be a general fiber of ϕ, by proposition 2.11 $(X_g, L'_{|X_g}) \cong (\mathbb{P}_n, \mathcal{O}(1))$. We consider an irreducible component $W \subset \mathrm{Chow}_{n,1}(X)$ which contains $[X_g]$. We note that W is projective of dimension $n+1$ and over W there exists a universal family of cycles $\pi : U \to W$ with the evaluation map $e : U \to X$. The map e is birational and either it is an isomorphism or it has a positive-dimensional fiber. In the former case, however, $(\phi : X \to Y) \cong (\pi : U \to W)$ and thus ϕ is equidimensional and we are done by Fujita's result. Thus, to arrive to a contradiction, we assume that e has a positive dimensional fiber. We note that, if $x_0 \in X$ is such that $\dim(e^{-1}(x_0)) > 0$, then by the property of functor $Chow$, we have $\dim\left(e(\pi^{-1}(\pi(e^{-1}(x_0))))\right) > n$, because otherwise all cycles in $\pi(e^{-1}(x_0))$ would be mapped to one n-dimensional cycle.

Let us consider an irreducible component V_U of $\mathrm{Hom}(\mathbb{P}_1, U)$ such that $f(\mathbb{P}_1)$ is a line in $X_g = \pi^{-1}([X_g]) \cong \mathbb{P}_n$ and f is isomorphism onto its image. The family V_U is unsplit and it maps, via the natural map $\tilde{e} : \mathrm{Hom}(\mathbb{P}_1, U) \to \mathrm{Hom}(\mathbb{P}_1, X)$, into an unsplit component V of $\mathrm{Hom}(\mathbb{P}_1, X)$ containing curves whose degree with respect to L is 1. Now, we note that $\mathrm{locus}(V_U, u) = \pi^{-1}(\pi(u)) \cong \mathbb{P}_n$ for a general $u \in U$, and because V_u is unsplit the equality $\mathrm{locus}(V_U, u) = \pi^{-1}(\pi(u))$ holds for any $u \in U$. Therefore $\mathrm{locus}(V, x_0) \supset e(\pi^{-1}(\pi(e^{-1}(x_0))))$ and since the dimension of the latter set is bigger than n we arrive to a contradiction to proposition 2.9. This concludes the proof of the first part of theorem 2.12.

Thus Y is smooth and we set $\mathcal{E} := \phi_*(L)$ so that $(X, L) = (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. We shall prove that $\mathcal{E} \cong T_Y$. By lemma 2.5 we have a map of extensions

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_X \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_{X/Y} \arrow[r] & \phi^*\mathcal{E}(-1) \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

and comparing it with the exact sequence defining $\Omega^1_{X/Y}$ we get an exact sequence

$$0 \to \phi^*(\Omega^1_Y) \to \mathcal{L} \to \phi^*\mathcal{E}(-1) \to 0$$

Now we combine the embedding $\phi^*(\Omega^1_Y) \to \mathcal{L}$ with the symplectic isomorphism $\omega : \mathcal{L} \cong \mathcal{L}^*(-1)$. By considering the restriction to fibers of ϕ we see that the induced map, coming from the twisted dual of the extension defining $\mathcal{L}$

$$\phi^*(\Omega^1_Y) \longrightarrow \mathcal{L} \xrightarrow{\omega} \mathcal{L}^*(-1) \longrightarrow \phi^*((\Omega^1_Y)^*)(-1)$$

is zero so we have an embedding of $\phi^*(\Omega^1_Y)$ into the kernel of $\mathcal{L}^*(-1) \to \phi^*((\Omega^1_Y)^*)(-1)$ which is $\phi^*(\mathcal{E}^*)$. The resulting map is of maximal rank so that we get $\phi^*(\Omega^1_Y) \cong \phi^*(\mathcal{E}^*)$ which implies $\mathcal{E} = T_Y$ and concludes the proof of theorem 2.12. □

2.6. **Contact structures on $\mathbb{P}(T_Y)$.** We now turn to contact structures on projectivized tangent bundles which we shall describe in more detail. At first, we consider contact varieties with more than one elementary contraction. It was known before that if X is Fano, then either $b_2(X) = 1$ or $X = \mathbb{P}(T_{\mathbb{P}_{n+1}})$. This was proved in [LS94, cor. 4.2] using Wiśniewski's classification of Fano manifolds. In fact, a stronger version of that result holds:

Proposition 2.13. *A projective contact manifold X admits one extremal ray contraction at most, unless $X \cong \mathbb{P}(T_{\mathbb{P}_{n+1}})$.*

Proof. If X has more than one extremal ray then, by the Mori cone theorem, the cone of curves of X has also a Mori face of dimension ≥ 2. This means that there exists a Mori contraction $\psi : X \to Y$ with $b_2(Y) \leq b_2(X) - 2$ which can be factored through at least two different elementary contractions ϕ_i as follows:

$$\begin{tikzcd} X \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] \arrow[dr, "\psi"] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y \end{tikzcd}$$

We claim that in this case Y is just a point so we are in the case of a Fano contact manifold with $b_2 \geq 2$ which, by the mentioned above result, is $\mathbb{P}(T_{\mathbb{P}_{n+1}})$. Indeed, otherwise $\dim Y > 0$ and we can apply proposition 2.11 to obtain $\psi^{-1}(y) \cong \mathbb{P}_n$, for a general $y \in Y$. But by lemma 2.10 $\psi^{-1}(y)$ must contain curves contracted by both ϕ_1 and ϕ_2, a contradiction, since $b_2(\mathbb{P}_n) = 1$ and thus all curves on $\psi^{-1}(y) \cong \mathbb{P}_n$ are numerically proportional. □

If $X = \mathbb{P}(T_{\mathbb{P}_{n+1}})$, then the contact structure on X is unique, see e.g. [LeB95, cor. 3.2]. Note also that $\operatorname{Aut}(\mathbb{P}(T_{\mathbb{P}_{n+1}}))/\operatorname{Aut}(\mathbb{P}_{n+1}) = \{1\}$. We will see that this is not a coincidence.

Let $\phi : X = \mathbb{P}(T_Y) \to Y$ be the projectivization of the tangent bundle on a compact complex manifold Y of dimension $n+1$, together with the relative ample sheaf $L = \mathcal{O}_{\mathbb{P}(T_Y)}(1)$. We have an exact sequence of bundles of twisted differentials

$$0 \to \phi^*(\Omega^1_Y)(1) \to \Omega^1_X(1) \to \Omega^1_{X/Y}(1) \to 0$$

We claim that the resulting map

$$H^0(X, \phi^*(\Omega^1_Y)(1)) \to H^0(X, \Omega^1_X(1))$$

is an isomorphism. This follows from observing that $H^0(X, \Omega^1_{X/Y}(1))$ is the kernel of the evaluation map

$$H^0(X, \phi^*(T_Y)) = H^0(X, \phi^*(\phi_*(\mathcal{O}_{\mathbb{P}(T_Y)}(1))) \to H^0(X, \mathcal{O}_{\mathbb{P}(T_Y)}(1))$$

appearing in the cohomology of twisted relative Euler sequence

$$0 \to \Omega^1_{X/Y}(1) \to \phi^*(T_Y) \to \mathcal{O}_{\mathbb{P}(T_Y)}(1) \to 0$$

On the other hand we have naturally, by push-forward,

$$H^0(X, \phi^*(\Omega^1_Y)(1)) = H^0(Y, \phi_*(\phi^*(\Omega^1_Y)(1))) = H^0(Y, \Omega^1_Y \otimes T_Y) = H^0(Y, \mathrm{End}(\Omega^1_Y)),$$

so sections of $\phi^*\Omega^1_Y(1)$ can be identified naturally with endomorphisms of Ω^1_Y.

Let us reveal the local nature of the above identification. In an analytic neighborhood U of a point $y \in Y$ we choose local (analytic !) coordinates (y^α) and, accordingly, we choose local vector fields ζ^β, where $0 \le \alpha, \beta \le n$, so that $\zeta^\alpha(y^\alpha) = 1$ and $\zeta^\beta(y^\alpha) = 0$ if $\alpha \ne \beta$. We note that ζ^β are also homogeneous coordinates in $\phi^{-1}(U)$ and, equivalently, generating sections of $\mathcal{O}_{\mathbb{P}(T_Y)}(1)_{|\phi^{-1}(U)}$. If $h = (h^{\beta\alpha}) \in \mathrm{End}(\Omega^1_Y)(U) \cong Mat((n+1) \times (n+1), \mathcal{O}(U))$ is an endomorphism of Ω^1_Y over U, represented in bases related to the choice of coordinates (y^α) as a $(n+1) \times (n+1)$ matrix then setting

$$\theta_h = \sum_{0 \le \alpha, \beta \le n} \zeta^\beta h^{\beta\alpha} dy^\alpha$$

we get a local presentation of h, related to the choice coordinates, a well as a well defined 1-form in $\Omega^1_X(1)_{|\phi^{-1}(U)}$.

Proposition 2.14. *The above defined identification*

$$H^0(Y, \mathrm{End}(\Omega^1_Y)) \ni h \mapsto \theta_h \in H^0(X, \Omega^1_X(1))$$

provides a bijection between automorphisms of Ω^1_Y and contact forms on $X = \mathbb{P}(T_Y)$.

Proof. It is enough to prove that $\theta_h \wedge (d\theta_h)^n \in H^0(X, K_X \otimes L^{n+1})$ does not vanish anywhere if and only if h is an automorphism. Since, however, $K_X \otimes L^{n+1} = \mathcal{O}_X$ it is enough to verify the non-vanishing at one point. This will be done using the local description provided above. First, we compute

$$d\theta_h = \sum_{0 \le \alpha, \beta \le n} \left[d\zeta^\beta \wedge (h^{\beta\alpha} dy^\alpha) + \zeta^\beta \cdot (dh^{\beta\alpha} \wedge dy^\alpha) \right]$$

and we see that terms of type $\zeta^\beta \cdot (dh^{\beta\alpha} \wedge dy^\alpha)$ can be ignored in the computation of $\theta_h \wedge (d\theta_h)^n$ because they do not contain differentials with respect to ζ^β. On the other hand we note that on the projective space $\mathbb{P}_n$ with homogeneous coordinates $[\zeta^0, \dots, \zeta^n]$ we can identify the form $\sum (-1)^k \cdot \zeta^k \cdot d\zeta^0 \wedge \cdots \hat{k} \cdots \wedge d\zeta^n$ with the unit section of $\Omega^n_{\mathbb{P}_n}(n+1) \cong \mathcal{O}_{\mathbb{P}_n}$. Moreover, if write $d\tilde{y}^\beta = \sum_\alpha h^{\beta\alpha} dy^\alpha$ then $d\tilde{y}^0 \wedge \cdots \wedge d\tilde{y}^n = \det(h) \cdot dy^0 \wedge \cdots \wedge dy^n$. So, taking all the above into account and remembering that the wedge product of degree 2 forms commutes we get

$$\begin{aligned}
&\theta_h \wedge (d\theta_h)^n \\
&= n! \cdot \sum_{k=0 \dots n} (\zeta^k \cdot d\tilde{y}^k) \wedge [(d\zeta^0 \wedge d\tilde{y}^0) \wedge \cdots \hat{k} \cdots \wedge (d\zeta^n \wedge d\tilde{y}^n)] \\
&= \pm n! \sum_{k=0 \dots n} (\zeta^k \cdot d\tilde{y}^k) \wedge (d\zeta^0 \wedge \cdots \hat{k} \cdots \wedge d\zeta^n) \wedge (d\tilde{y}^0 \wedge \cdots \hat{k} \cdots \wedge d\tilde{y}^n) \\
&= \pm n! \sum_{k=0 \dots n} (-1)^k \cdot \zeta^k \cdot (d\zeta^0 \wedge \cdots \hat{k} \cdots \wedge d\zeta^n) \wedge (d\tilde{y}^0 \wedge \cdots \wedge d\tilde{y}^n) \\
&= \pm n! \cdot \det(h) \cdot \left[\sum_k (-1)^k \cdot \zeta^k \cdot (d\zeta^0 \wedge \cdots \hat{k} \cdots \wedge d\zeta^n) \right] \wedge (dy^0 \wedge \cdots \wedge dy^n)
\end{aligned}$$

and the last expression is non-zero if and only if $\det(h) \ne 0$. This concludes the proof of proposition 2.14. $\square$

3. CONTACT MANIFOLDS WHERE K_X IS NEF

In this section we turn to the case where K_X is nef. It is shown in [Dru98] that a projective contact manifold always has Kodaira dimension $\kappa(X) = -\infty$. The abundance conjecture predicts that this is incompatible with K_X nef. However, since the conjecture is known only in dimension ≤ 3 and completely open in higher dimensions, we have to consider this possibility here.

As a partial result, we show in theorem 3.1 that K_X is not nef if X has more than one contact structure, and in proposition 3.5 that K_X is not nef if a certain stability property of T_X holds.

Proposition 3.1. *Let X be a projective manifold admitting at least two contact structures. Then the canonical bundle K_X is not nef and thus either $X = \mathbb{P}(T_Y)$ or X is Fano and $b_2(X) = 1$.*

Proof. Let

$$0 \to F \to T_X \to L \to 0 \tag{3.1}$$

and

$$0 \to F' \to T_X \to L' \to 0$$

be two different contact structures on X. Since $L^{n+1} = (L')^{n+1} = -K_X$, the line bundle $L - L'$ is torsion and, after passing to a finite étale cover, we may assume that $L = L'$.

The map $F \to L'$ yields a non-zero element $v \in H^0(F^* \otimes L') = H^0(F^* \otimes L) \cong H^0(F) \subset H^0(T_X)$, i.e. we obtain a vector field. If v has zeroes, then X is uniruled and we see that K_X cannot be nef. Thus, we assume that v has no zeroes. In this situation [Lie78, thm. 3.13] asserts that —after another étale cover, if necessary— $X \cong T \times Y$, where T is a torus and $v \in \pi_1^*(H^0(T_T)) \subset H^0(F)$. Here $\pi_1 : X \to T$ and $\pi_2 : X \to Y$ are the natural projections. It follows from the adjunction formula that the restriction of L to π_2-fibers is torsion. Thus, again taking covers, we may assume that $L = \pi_2^*(L_Y)$ with $L_Y \in \operatorname{Pic}(Y)$. Let $U \subset Y$ denote an open set such that $L_Y|_U = \mathcal{O}_U$. Let

$$\theta \in H^0(\pi_2^{-1}(U), \Omega_X^1 \otimes L) \cong H^0(\pi_2^{-1}(U), \Omega_X^1) \cong H^0(\pi_1^*(\Omega_T^1)) \oplus H^0(\pi_2^*(\Omega_Y^1))$$

be the contact form associated with sequence 3.1 and $\theta = \theta_T \oplus \theta_Y$ be the direct sum decomposition. By assumption, the following expression is not zero:

$$\theta \wedge (d\theta)^{\wedge n} = (\theta_T + \theta_Y) \wedge (d\theta_T + d\theta_Y)^{\wedge n} = \theta_T \wedge (d\theta_Y)^{\wedge n}.$$

This, however, is absurd: first, it follows from $(d\theta_Y)^{\wedge n} \neq 0$ that $\dim Y = 2n$, $\dim T = 1$. It then follows from $\theta(v) = 0$ that $\theta_T = 0$. A contradiction. □

As a first step towards the proof of proposition 3.5, the succeeding result asserts that in our situation K_X^2 is zero. For this we do not actually need that X has a contact structure and consider a more general situation:

Proposition 3.2. *Let X be a projective manifold. Let $L^* \subset \Omega_X^1$ be a locally free subsheaf of rank 1 with $\alpha L^* \equiv K_X$ for some positive rational number α. If K_X is additionally nef, then $K_X^2 = 0$.*

The notation L^* might seem to be slightly awkward at first. We use it to be consistent with the notation introduced in section 2.

Proof. Consider the smooth surface S cut out by general hyperplane sections

$$S := H_1 \cap \ldots \cap H_{\dim X - 2}.$$

Since L^* is nef, there is no morphism $L^*|_S \to N_{S|X}^*$, and we obtain an injection $L^* \to \Omega_S^1$. By Bogomolov's well-known theorem (see [Bog79, thm. on p. 501]), $L^*|_S$ cannot be big. Thus $(L^*)^2.S = 0$ which translates into

$$K_X^2.H_1 \ldots H_{\dim X - 2} = 0.$$

We claim that if D is any nef divisor with $D^2.H_1 \dots H_{\dim X-2} = 0$. for generic ample divisors H_i, then $D^2 \equiv 0$. If $\dim X = 3$, the claim is obvious. If $\dim X \geq 5$, then we can apply the Lefschetz theorem and reduce to the case where $\dim X = 4$ by taking suitable hyperplane sections.

If $\dim X = 4$ and H is any ample divisor, then $D + \epsilon H$ is an ample $\mathbb{R}$-divisor for all $\epsilon > 0$. Consequently, since $(D + \epsilon H)^2.H_1 \in \overline{NE(X)}$, we have $D^2.H_1 \in \overline{NE(X)}$, and we conclude that $D^2.H_1 \equiv 0$. Now apply the Hodge index theorem in the form of [Har77, App. A, 5.2] with $Y := D^2$ and $H := H_1$. Assuming that $D \not\equiv 0$ that theorem yields that $D^4 > 0$. But

$$D^4 = \lim_{\epsilon \to 0} D^2. \underbrace{(D + \epsilon H_1).(D + \epsilon H_2)}_{\text{two gen. ample divisors}} = 0,$$

a contradiction. $\square$

The proof the preceding proposition gives rise to the following

Problem 3.3. Let S be a smooth projective surface and $L^* \subset \Omega^1_X$ a numerically effective locally free subsheaf of rank one with $(L^*)^2 = 0$ and $L^* \not\equiv 0$. Can anything be said about the structure of S? Observe that if $L^* \subset \Omega^1_X$ is a subbundle away from a finite set (e.g. if X is a contact manifold), then $L^*|_S \subset \Omega^1_S$ is a subbundle away from a finite set as well.

Corollary 3.4. *In the setting of proposition 3.2, the tangent bundle T_X is semistable with respect to $(K_X, H^{\dim X-2})$, where H is any ample line bundle. This means*

$$\frac{1}{r} c_1(\mathcal{S}) \cdot K_X \cdot H^{\dim X-2} \leq \frac{1}{\dim X} c_1(X) \cdot K_X \cdot H^{\dim X-2}$$

for all coherent subsheaves $\mathcal{S} \subset T_X$ of any rank $r > 0$.

Proof. Since $K_X^2 \equiv 0$, this follows from [Eno87]. $\square$

Unfortunately it is not always true that T_X is also semistable with respect to $(K_X + \epsilon H, H^{m-2})$ for some ample H and some small number $\epsilon > 0$. Counterexamples are e.g. provided by products of elliptic curves with curves of genus $g \geq 2$. On the other hand, a weak consequence of this semistability property would already imply the assertion that K_X is not nef:

Theorem 3.5. *In the setting of proposition 3.2, if there exists an ample bundle $H \in \operatorname{Pic}(X)$ and a number $\epsilon > 0$ such that L^* does not destabilize Ω^1_X with respect to $(K_X + \epsilon H, H^{\dim X-2})$ and if $0 < \alpha < \dim X$, then either K_X is not nef or $K_X \equiv 0$.*

Proof. The assertion on the destabilization can be expressed as

$$c_1(L^*) \cdot (K_X + \epsilon H) \cdot H^{\dim X-2} \leq \frac{1}{\dim X} K_X \cdot (K_X + \epsilon H) \cdot H^{\dim X-2}.$$

Using $K_X^2 = 0$ and $K_X = \alpha L^*$, the inequality becomes

$$\frac{1}{\alpha} K_X \cdot H^{\dim X-1} \leq \frac{1}{\dim X} K_X \cdot H^{\dim X-1} \tag{3.2}$$

If we now assume that K_X is nef, then inequality (3.2) implies $K_X \cdot H^{\dim X-1} = 0$, which is equivalent to $K_X \equiv 0$. $\square$

Therefore it remains to consider the case that L^* destabilizes T_X for all polarizations $(K_X + \epsilon H, H^{\dim X-2})$. One might hope to derive some geometric consequences from this unstability.

Remark that all considerations of the present section apply in particular to contact manifolds where $\dim X = 2n+1$ and $\alpha = n+1$. Thus, if the unstable case could be handled, then the canonical bundle K_X of a projective contact manifold X would never be nef. This is because the assumption $K_X \equiv 0$ contradicts a result of Ye [Ye94, lem. 1].

4. MANIFOLDS WITH NEF SUBSHEAVES IN Ω^1_X

The setup of proposition 3.2 seems to be of independent interest. The aim of the present section is to give a description of these varieties.

Assumptions 4.1. Let X be a projective manifold and $L^* \subset \Omega^1_X$ be a locally free nef subsheaf of rank one. Assume that there is a positive rational number α such that $\alpha L^* = K_X$.

Recall the well-known result of Bogomolov [Bog79] which implies that in this setting $\kappa(X) \leq 1$.

As a first result we obtain:

Proposition 4.2. *If the Kodaira dimension $\kappa(X) \geq 0$, then $K_X \equiv 0$ or $\alpha \geq 1$.*

Proof. We argue by absurdity: assume that $K_X \not\equiv 0$ and $\alpha < 1$. The inclusion $L^* \to \Omega^1_X$ gives a non-zero element

$$\theta \in H^0(\Omega^1_X \otimes L) = H^0\left(\wedge^{\dim X - 1} T_X \otimes \left(1 - \frac{1}{\alpha}\right) K_X\right)$$

Suppose that $\alpha < 1$. Then $(1 - \frac{1}{\alpha}) < 0$ and thus we can find positive integers m, p such that

$$H^0(\underbrace{S^m(\wedge^{\dim X - 1} T_X) \otimes \mathcal{O}(-pK_X)}_{=:E}) \neq 0.$$

In order to derive a contradiction, recall the result of Miyaoka ([Miy87, cor. 8.6]; note that X cannot be uniruled) that $\Omega^1_X|_C$ is nef for a sufficiently general curve $C \subset X$ cut out by general hyperplane sections of large degree. Remark that $E^*|_C$ is nef, too. We may assume without loss of generality that p is big enough so that pK_X has a section with zeroes, and so does $E|_C$. See [CP91, prop. 1.2] for a list of basic properties of nef vector bundles which shows that this is impossible. $\square$

4.1. The case where $\kappa(X) = 1$.

In this case K_X is semi-ample and we give a description of L^* in terms of the Kodaira-Iitaka map.

Theorem 4.3. *Under the assumptions 4.1, if $\kappa(X) = 1$, then K_X is semi-ample, i.e. some multiple is generated by global sections. Let $f : X \to C$ be the Iitaka fibration and B denote the divisor part of the zeroes of the natural map $f^*(\Omega^1_C) \to \Omega^1_X$. Then there exists an effective divisor $D \in \mathrm{Div}(X)$ such that $L^* = f^*(\Omega^1_C) \otimes \mathcal{O}_X(B - D)$.*

Furthermore, if p is chosen such that $pK_X \in f^*(\mathrm{Pic}(C))$ and such that $p\alpha$ is an integer, then

$$p(\alpha - 1)K_C = f_*(pK_{X|C}) + f_*(p\alpha(D - B)).$$

Proof. Since $K_X \not\equiv 0$ and $K_X^2 = 0$ by proposition 3.2, the numerical dimension $\nu(X) = 1$. In this setting [Kaw85, thm. 1.1] proves that K_X is semi-ample, i.e. that a sufficiently high multiple of K_X is globally generated. By [Bog79, lem. 12.7], the canonical morphism

$$L^* \to \Omega^1_{X|C}$$

is generically 0. Let B denote the divisor part of the zeroes of $f^*(\Omega^1_C) \to \Omega^1_X$. Then we obtain an exact sequence

$$0 \to f^*(\Omega^1_C) \otimes \mathcal{O}_X(B) \to \Omega^1_X \to \tilde{\Omega}^1_{X|C} \to 0$$

where the cokernel $\tilde{\Omega}^1_{X|C}$ is torsion free. Since $\Omega^1_{X|C} = \tilde{\Omega}^1_{X|C}$ away from a closed subvariety, the induced map

$$L^* \to \tilde{\Omega}^1_{X|C}$$

vanishes everywhere. Hence there exists an effective divisor D such that

$$L^* \subset f^*(\Omega^1_C) \otimes \mathcal{O}_X(B) \quad \text{i.e.} \quad L^* = f^*(\Omega^1_C) \otimes \mathcal{O}_X(B - D).$$

The last equation is obvious. $\square$

4.2. **The case where $\kappa(X) = 0$.** We now investigate the more subtle case $\kappa(X) = 0$. We pose the following

Conjecture 4.4. *Under the assumptions 4.1, if $\kappa(X) = 0$, then $K_X \equiv 0$. Hence (see* [Bea83]*) there exists a finite étale cover $\gamma : \tilde{X} \to X$ such that $\gamma^*(L^*) = \mathcal{O}_{\tilde{X}}$ and $\tilde{X} = A \times Y$ where A is Abelian and Y is simply connected.*

We will prove the conjecture in a number of cases, in particular if $\dim X \leq 4$ or if the well-known Conjecture K holds (see [Mor87, sect. 10] for a detailed discussion). Recall that the Albanese map of a projective manifold X with $\kappa(X) = 0$ is surjective and has connected fibers [Kaw81].

Conjecture 4.5 (Ueno's Conjecture K)**.** *If X is a nonsingular projective variety with $\kappa(X) = 0$, then the Albanese map is birational to an étale fiber bundle over* $\mathrm{Alb}(X)$ *which is trivialized by an étale base change.*

It is known that Conjecture K holds if $q(X) \geq \dim X - 2$.

The proof requires two technical lemmata. The first is a characterization of pull-back divisors. It appears implicitly in [Kaw85, p. 571].

Lemma 4.6 (Kawamata's pull-back lemma)**.** *Let $f : X \to Y$ be an equidimensional surjective projective morphism with connected fibers between quasi-projective $\mathbb{Q}$-factorial varieties and let $D \in \mathbb{Q}\,\mathrm{Div}(X)$ be an f-nef $\mathbb{Q}$-divisor such that $f(\mathrm{Supp}(D)) \neq Y$. Then there exists a number $m \in \mathbb{N}^+$ and a divisor $A \in \mathrm{Div}(Y)$ such that $mD = f^*(A)$.*

The next lemma is very similar in nature, and also the proof is very kind. We include it here for lack of an adequate reference.

Lemma 4.7. *Let $f : X \to Y$ be a surjective projective morphism with connected fibers between quasi-projective $\mathbb{Q}$-factorial varieties and let $D \in \mathbb{Q}\,\mathrm{Div}(X)$ be an f-nef $\mathbb{Q}$-divisor which is of the form*

$$D = f^*(A) + \sum \lambda_i E_i$$

where $A \in \mathbb{Q}\,\mathrm{Div}(Y)$ and $\mathrm{codim}_Y f(E_i) \geq 2$. Then $\lambda_i \leq 0$ for all i.

Proof. Suppose that this was not the case. Choose j such that $\lambda_j > 0$ and such that $f(E_j)$ is of maximal dimension. As the lemma is formulated for quasi-projective varieties, in order to derive a contradiction, we may assume without loss of generality that $f(E_i) = f(E_j)$ for all i.

Claim: There exists a divisor $B \in \mathbb{Q}\,\mathrm{Div}(Y)$ and a number $m \in \mathbb{N}^+$ such that the $\mathbb{Q}$-divisor $M := mD - f^*(B)$ satisfies the following conditions

1. $-M$ is effective
2. there exists a component F_i of $F := f^{-1}f(E_j)$ which is not contained in M
3. $F \cap \mathrm{Supp}(M) \neq \emptyset$

Application of the claim: If the claim is true, then we can always find a curve $C \subset F$ such that $C \not\subset \mathrm{Supp}(M)$, $C \cap \mathrm{Supp}(M) \neq \emptyset$ and $f(C) = (*)$. Since $-M$ is effective, we have $C.M < 0$, contradicting the assumed f-nefness.

Proof of the claim: choose a very ample effective Cartier-divisor $H \in \mathrm{Div}(Y)$ such that

- the effective part of A is contained in H and
- $f(E_i) \subset \mathrm{Supp}(H)$.

If $M := D$ satisfies the requirements of the claim already, stop here. Otherwise, set

$$D' := \underbrace{(\text{mult. of } E_j \text{ in } f^*(H))}_{\text{positive}}.D - \underbrace{(\text{mult. of } E_j \text{ in D})}_{\text{positive}}. \quad \underbrace{f^*(H)}_{\text{cont. } E_i \text{ with pos. mult.}}$$

by choice of H and by choice of the coefficients, D' satisfies conditions (2) and (3) of the claim. If $-D'$ is effective, then we are finished already.

If $-D'$ is not effective, note that the positive part of D' is supported on F only. Repeat the above procedure with a new number j. Note that after finitely many steps the divisor must become anti-effective. This finishes the proof. $\square$

We will now show that Conjecture K implies our conjecture 4.4.

Theorem 4.8. *If conjecture K holds, then conjecture 4.4 holds as well.*

In order to clarify the structure of the proof, we single out the case where $H^0(X, L^*) \neq 0$. More precisely, we consider the weakened

Assumptions 4.9. Let X be a projective manifold and $L^* \subset \Omega^1_X$ be a locally free nef subsheaf of rank one. Assume that $\kappa(X) = 0$ and that there is a positive rational number α such that $\alpha L^* \subset K_X$ is a subsheaf. Assume furthermore that $H^0(X, L^*) \neq 0$.

Lemma 4.10. *Under the weakened assumptions 4.9, if Conjecture K holds for X, then conjecture 4.4 holds as well, i.e. $L^* \equiv 0$.*

Proof. In inequality $H^0(X, L^*) \neq 0$ directly implies that $q(X) > 0$, i.e. that the Albanese map $f : X \to A = \mathrm{Alb}(X)$ is not trivial.

In this setting, Conjecture K yields a diagram as follows:

$$\begin{tikzcd}
 & X_{dom} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & & & \\
X \arrow[rr, dashed] \arrow[d, "f"'] & & X' \arrow[d, "f'", "\text{étale fiber bundle}"'] & F \times \tilde{A} \arrow[l, "\Gamma", "\text{étale, finite}"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & F \\
A \arrow[rr, equal] & & A & \tilde{A} \arrow[l, "\gamma", "\text{étale, finite}"'] &
\end{tikzcd}$$

where π and π' are birational morphisms. Let E be the zero-set of a section of L^*; i.e. $L^* = \mathcal{O}(E)$. We will show that $E = 0$.

Claim 1: The divisor E is supported on f-fibers, i.e. $f(\mathrm{Supp}(E)) \neq A$.

Proof of claim 1: the inclusion $L^* \subset \Omega^1_X$ yields an element

$$\theta \in H^0(\Omega^1_X(-E)).$$

Since $\omega = f^*(\eta)$, we conclude that

$$f(E) \subset \mathrm{Sing}(f),$$

where $\mathrm{Sing}(f)$ is the set of all $y \in A$ such that $f^{-1}(y)$ is singular. In particular, E does not meet the general fiber of f. This shows claim 1.

As a next step in the proof of lemma 4.10, we set

$$A^0 := \{a \in A | \dim f^{-1}(a) = \dim X - \dim A\} \quad \text{and} \quad X^0 := f^{-1}(A^0)$$

and show

Claim 2: the divisor E does not intersect X^0, i.e. $\mathrm{Supp}(E) \cap X^0 = \emptyset$.

Proof of claim 2: by Kawamata's pull-back lemma 4.6 there exists a number $m \in \mathbb{N}^+$ such that $mE|_{X^0} \in f^*(D|_{A^0})$ for some $D \in \operatorname{Div}(X)$. This implies

$$\overline{f^{-1}(D \cap A^0)} \subset \operatorname{Supp}(E).$$

Now consider X_{dom}. Since X is smooth, we can find a $\mathbb{Q}$-divisor for the canonical bundle $\omega_{X_{dom}}$ of the form

$$K_{X_{dom}} = \pi^*(\underbrace{K_X}_{\alpha E + \text{(effective)}}) + \sum \lambda_i B_i$$

where the B_i are π-exceptional divisors and $\lambda_i \in \mathbb{Q}^+$. In particular, we have

$$\overline{(\pi \circ f)^{-1}(D \cap X^0)} \subset \operatorname{Supp}(\pi^*(E)) \subset \operatorname{Supp}(K_{X_{dom}}).$$

Since $\pi \circ f = \pi' \circ f'$ and $K_{X'} := (\pi')_*(K_{X_{dom}})$ is a $\mathbb{Q}$-divisor for the anticanonical bundle $\omega_{X'}$, we find

$$\overline{(f')^{-1}(D \cap X^0)} \subset \operatorname{Supp}(K_{X'}).$$

We derive a contradiction from the last inequality by noting that

$$(\pi_2 \circ \gamma)^{-1}(D) \subset \operatorname{Supp}(\underbrace{K_{F \times \tilde{A}}}_{= \Gamma^*(K_{X'})}).$$

On the other hand, it follows from the adjunction formula that every effective $\mathbb{Q}$-divisor for the canonical bundle $\omega_{F \times \tilde{A}}$ must be contained in $\pi_1^*(\mathbb{Q}\operatorname{Div}(F))$. This shows claim 2.

Application of claims 1 and 2: we know that E is an effective and nef divisor satisfying $\operatorname{codim}_A(f(\operatorname{Supp}(E))) \geq 2$. By lemma 4.7 this is possible if and only if $E = 0$. ☐

The preceding lemma enables us to finish the proof of theorem 4.8 by reducing to the modified weakened assumptions 4.9. Since we wish to re-use the same argumentation later, we formulate a technical reduction lemma:

Lemma 4.11 (Reduction Lemma)**.** *If $L^* \equiv 0$ holds true for all varieties of a given dimension d satisfying the weakened assumptions 4.9, then conjecture 4.4 holds for all varieties of dimension d.*

Proof. Let X be a variety as in conjecture 4.4 and assume that X is of dimension d. If $H^0(X, L^*) \neq 0$, then we can stop here.

Otherwise, we find a minimal positive integer m and an effective divisor E such that $(L^*)^m = \mathcal{O}_X(E)$. We build a sequence of morphisms

$$\tilde{Y} \xrightarrow[\text{desing.}]{\sigma} Y \xrightarrow[\text{cyclic cover}]{\gamma} \hat{X} \xrightarrow[\text{desing. of } E]{\pi} X$$

as follows: Let $\pi : \hat{X} \to X$ be a sequence of blow-ups with smooth centers such that $\operatorname{Supp}(\pi^*(E))$ has only normal crossings. Set $\hat{E} = \pi^*(E)$ and $\hat{L} = \pi^*(L)$ and let $f : Y \to \hat{X}$ be the cyclic covering (followed by normalization) associated with the section

$$s \in H^0(X, (L^*)^m)$$

which defines $\hat{E}$; see e.g. [EV92, 3.5] for a more detailed description of this construction. By [EV92, 3.15], Y is irreducible, étale over $\hat{X} \setminus \operatorname{Supp}(\hat{E})$ and smooth over $\hat{X} \setminus \operatorname{Sing}(\operatorname{Supp}(\hat{E}))$. Moreover

$$H^0(Y, \gamma^*(\hat{L}^*)) \neq 0.$$

Let $\sigma : \tilde{Y} \to Y$ be a desingularization. We will show:

Claim: $\kappa(\tilde{Y}) = 0$.

Application of the claim: if the claim holds true, we conclude as follows: let $\tilde{L} = (\sigma \circ \gamma)^*(\hat{L})$. Then $H^0(\tilde{X}, \tilde{L}^*) \neq 0$, and furthermore $\tilde{L}^* \subset \Omega^1_{\tilde{Y}}$ by virtue of the canonical morphism

$$\underbrace{(\sigma \circ \gamma \circ \pi)}_{=:\Gamma}{}^*(\Omega^1_X) \to \Omega^1_{\tilde{Y}}.$$

Similarly, since X is smooth, $\Gamma^* K_X \subset K_{\tilde{Y}}$. Thus, lemma 4.10 applies to $\tilde{Y}$, showing that $\Gamma^*(L^*) \equiv 0$ so that L^* was numerically trivial in the first place. This shows the lemma and thus finishes the proof of theorem 4.8.

Proof of the claim: Since Γ is an étale cover away from E, we can find a divisor for the canonical bundle $\omega_{\tilde{Y}}$ which is of the form

$$K_{\tilde{Y}} = \Gamma^*(\alpha E) + D$$

where D is effective and $\Gamma(D) \subset E$. This already implies that there is a number $k \in \mathbb{N}^+$ such that $(k\Gamma^*(E) - K_{\tilde{Y}})$ is effective. Thus $\kappa(K_{\tilde{Y}}) \leq \kappa(\Gamma^*(E)) = 0$, and the claim is shown. □

Thus ends the proof of theorem 4.8.

Finally we show conjecture 4.4 in the case where $\dim X \leq 4$.

Theorem 4.12. *Under the assumptions 4.1, if $\kappa(X) = 0$ and $\dim X \leq 4$, then $K_X \equiv 0$.*

Proof. Using the reduction lemma 4.11, it is sufficient to consider the weakened setting 4.9: let X be as in 4.9. At first we argue exactly as in the proof of lemma 4.10: the Albanese map $f : X \to A = \mathrm{Alb}(X)$ is surjective and has connected fibers and we have $f(\mathrm{Supp}(E)) \neq A$, where E is the zero-divisor of the section in L^*. Furthermore we have $\kappa(X, E) = 0$.

Recall that Conjecture K holds if $q(X) \geq \dim X - 2$; see [Mor87, p. 316]. By lemma 4.10 we are finished in these cases. It remains to consider the case where $q(X) = 1$.

Because f is equidimensional in this case, Kawamata's pull-back lemma 4.6 applies: there is a number $m \in \mathbb{N}^+$ such that $mE = f^*(D)$. It follows immediately that $\kappa(A, D) = 0$. Since A is a torus, this is possible if and only if $D = 0$. □

Remark 4.13. Actually, the proof of lemma 4.11 and theorem 4.12 show that conjecture 4.4 holds if $q(X) \geq \dim X - 2$ or if $H^0(X, L^*) \neq 0$ and $q(X) = 1$. For the first statement, note that $q(X)$ increases when passing to a cover.

REFERENCES

[Ati57] M. Atiyah. Complex analytic connections in fibre bundles. *Trans. Am. Math. Soc.*, 85:181–207, 1957.

[Bea83] A. Beauville. Variétés Kählériennes dont la première classe de Chern est nulle. *J. Diff. Geom.*, 18:755–782, 1983.

[Bea98] A. Beauville. Fano contact manifolds and nilpotent orbits. *Comm. Math. Helv.*, 73(4):566–583, 1998.

[Bea99] A. Beauville. Riemannian holonomy and Algebraic Geometry. Duke/alg-geom Preprint 9902110, 1999.

[Bog79] F. Bogomolov. Holomorphic tensors and vector bundles on projective varieties. *Math. USSR Izv.*, 13:499–555, 1979.

[BS95] M. Beltrametti and A. Sommese. *The Adjunction Theory of Complex Projective Varieties*. de Gruyter, 1995.

[CP91] F. Campana and T. Peternell. Projective manifolds whose tangent bundles are numerically effective. *Math. Ann.*, 289:169–187, 1991.

[Dru98] S. Druel. Contact structures on algebraic 5-dimensional manifolds. C.R. Acad. Sci.Paris, 327:365-368, 1998.

[Eno87] I. Enoki. Stability and negativity for tangent bundles of minimal Kähler spaces. In T. Sunada, editor, *Geometry and Analysis on Manifolds. Proceedings, Katata-Kyoto*, volume 1339 of *Lecture Notes in Mathematics*, pages 118–127. Springer, 1987.

[EV92] H. Esnault and E. Viehweg. *Lectures on Vanishing Theorems*. Birkhäuser Verlag, 1992.

[Fuj85] T. Fujita. On Polarized Manifolds Whose Adjoint Bundles Are Not Semipositive. In T. Oda, editor, *Advanced Studies in Pure Mathematics 10*, pages 167–178. North-Holland publishing company, 1985.

[Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[Kaw81] Y. Kawamata. Characterization of Abelian varieties. *Compositio Math.*, 43:253–276, 1981.
[Kaw85] Y. Kawamata. Pluricanonical systems on minimal algebraic varieties. *Inv. Math.*, 79:567–588, 1985.
[Kol96] J. Kollár. *Rational Curves on Algebraic Varieties*. volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1996.
[KM98] J. Kollár and S. Mori. Birational Geometry of algebraic varieties. volume 134 of *Cambridge Tracts in Math*. Cambridge University Press, 1998.
[LeB95] C. LeBrun. Fano manifolds, contact structures and quaternionic geometry. *Int. Journ. of Math.*, 6(3):419–437, 1995.
[LS94] C. LeBrun and S. Salamon. Strong rigidity of positive quaternion-Kähler manifolds. *Inv. Math.*, 118(1):109–132, 1994.
[Lie78] D. Liebermann. Compactness of the Chow scheme: Applications to automorphisms and deformations of Kähler manifolds. In Francois Norguet, editor, *Fonctions de Plusieurs Variables Complexes III*, number 670 in Lecture Notes in Mathematics, pages 140–186. Springer, 1978.
[Miy87] Y. Miyaoka. Deformation of a morphism along a foliation. In S. Bloch, editor, *Algebraic Geometry*, volume 46(1) of *Proceedings of Symposia in Pure Mathematics*, pages 245–268, Providence, Rhode Island, 1987. American Mathematical Society.
[Mor87] S. Mori. Classification of higher-dimensional varieties. In *Proceedings of Symposia in Pure Mathematics*, volume 46, of *Proceedings of Symposia in Pure Mathematics*, pages 269–331, Providence, Rhode Island, 1987. American Mathematical Society.
[Wah83] J. Wahl. A cohomological characterization of $\mathbb{P}_n$. *Inv. Math.*, 72:315–322, 1983.
[Ye94] Y.-G. Ye. A note on complex projective threefolds admitting holomorphic contact structures. *Inv. Math.*, 115:311–314, 1994.

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